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# NONREFLEXIVE SPACES OF TYPE 2

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ABSTRACT
The nonreflexive and uniformly nonoctahedral spaces $X_\rho$ are known to be of type $p$ if $1 \leqq p < 2$ and $\rho$ is sufficiently large. It is shown that $X_\rho$ is of type 2 if $\rho > 2$.

A Banach space $X$ is of type $p$ if there is a constant $C$ such that, for any choice of $\{x^i: 1 \leqq i \leqq n\}$ in $X$, we have

(1) $$2^{-n}\sum_\sigma \left\| \left(\sum_{i=1}^n \pm x^i\right)_\sigma \right\| \leqq C\left(\sum_{i=1}^n \|x^i\|^p\right)^{1/p},$$

where the summation is over all sequences $\sigma$ of $n$ signs. It will be shown that there are nonreflexive spaces of type 2. The question of a relation between reflexivity and "type 2" was raised in [1, p. 646]. Davis and Lindenstrauss showed that for each $p < 2$ there is a nonreflexive space of type $p$ [2, theor. 3, p. 193]. Kwapień showed that a Banach space $X$ is isomorphic to a Hilbert space if it is of type 2 and cotype 2 [6, theor. 1]. Pisier established a stronger result [10, prop., p. 348], which implies that $X$ is super-reflexive if $X$ is of type 2 and there is a sequence $\{C_n\}$ such that $\lim_{n\to\infty} C_n \ln n = \infty$ and, for each $n$ and any choice of $\{x^i: 1 \leqq i \leqq n\}$ in $X$,

$$2^{-n}\sum_\sigma \left\| \left(\sum_{i=1}^n \pm x^i\right)_\sigma \right\| \geqq C_n\left(\sum_{i=1}^n \|x^i\|^2\right)^{1/2}.$$

The first nonreflexive spaces known to have type greater than 1 were the uniformly nonoctahedral spaces given in [4]. The definition of these spaces was improved considerably in [5, p. 104]. With only a rather superficial change, this is the definition to be used here. The change yields a minor improvement in the coefficient of $n^{\frac{1}{2}}$ in inequality (18).

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It is an unpublished result of G. Pisier that a Banach space is of type 2 if it is of "equal-norm type 2", i.e., if (1) is satisfied when $p = 2$ and $\|x^i\| = 1$ for each $i$. Using equal norms is of vital importance for the methods of this paper. Therefore we include the following proposition.

PROPOSITION (Pisier). *A Banach space $X$ is of type 2 if there is a constant $C$ such that, for any choice of $\{x^i: 1 \leqq i \leqq n\}$ in $X$ with $\|x^i\| = 1$ for each $i$,*

$$2^{-n}\sum_{\sigma}\left\|\left(\sum_{i=1}^{n} \pm x^i\right)_{\sigma}\right\| \leqq Cn^{\frac{1}{2}}.$$

PROOF. If follows from Proposition 5 of [9] that if $\{g_i\}$ are independent Gaussian random variables with means 0, then $X$ is of type 2 if and only if there is a constant $D$ such that, for any choice of $\{x^i: 1 \leqq i \leqq n\}$ in $X$,

$$\int_{-\infty}^{\infty}\left\|\sum_{i=1}^{n} g_i(t)x^i\right\|dt \leqq D\left(\sum_{i=1}^{n}\|x^i\|^2\right)^{1/2}.$$

If (1) is satisfied for equal norms, and $\{g_i\}$ are independent normalized Gaussian random variables, then by repeating $x^i$'s it follows from the central limit theorem that

$$\int_{-\infty}^{\infty}\left\|\sum_{i=1}^{n} g_i(t)x^i\right\|dt \leqq Cn^{\frac{1}{2}},$$

if $\|x^i\| = 1$ for $1 \leqq i \leqq n$. Now suppose that $\{x^i: 1 \leqq i \leqq n\}$ are given and that $\|x^i\| = p_i/N$ for each $i$. Let $x^i_j = Nx^i/p_i$ for $1 \leqq j \leqq p_i^2$, and let $\{g_{ij}\}$ be independent normalized Gaussian random variables. Then

$$\int_{-\infty}^{\infty}\left\|\sum_{i,j} g_{ij}(t)x^i_j\right\|dt \leqq C\left(\sum_{i=1}^{n} p_i^2\right)^{1/2}.$$

The coefficient of $x^i$ in the integrand is $N(\sum_{j=1}^{p_i^2} g_{ij})/p_i$. Thus if $G_i = (\sum_{j=1}^{p_i^2} g_{ij})/p_i$, then $\{G_i\}$ are independent normalized Gaussian random variables and

$$\int_{-\infty}^{\infty}\left\|\sum_{i=1}^{n} G_i(t)x^i\right\|dt \leqq \frac{C}{N}\left(\sum_{i=1}^{n} p_i^2\right)^{1/2} = C\left(\sum_{i=1}^{n}\|x^i\|^2\right)^{1/2},$$

which implies $X$ is of type 2.

By a *bump* we mean any function which is equal to some nonzero constant on an interval of positive integers, and is equal to 0 at all other positive integers. This constant is the *altitude* of the bump. Two bumps are said to be *disjoint* if the intervals on which they are nonzero are disjoint. For $1 < \rho < \infty$, define a functional $[\![\ ]\!]$ on the set of sequences with finite support, letting

$$\llbracket x \rrbracket^{\rho} = \inf\left\{\sum_{\mu=1}^{m} r_{\mu}\left[\left(\sum_{k=\mu}^{m} h_k\right)^{\rho} - \left(\sum_{k=\mu+1}^{m} h_k\right)^{\rho}\right]\right\}, \tag{2}$$

where $x = \sum_{\mu=1}^{m} x^{\mu}$ and each $x^{\mu}$ is the sum of $r_{\mu}$ disjoint bumps whose altitudes have absolute value $h_{\mu}$ (in [5] it was required that the bumps in $r_{\mu}$ all have the same altitude). The functional $\llbracket\ \rrbracket$ does not satisfy the triangle inequality, so we let

$$\|x\| = \inf\left\{\sum_{k=1}^{n} \llbracket x^k \rrbracket : x = \sum_{k=1}^{n} x^k\right\}.$$

The completion with respect to this norm of the space of finitely supported sequences will be called $X_{\rho}$. As observed in [4, p. 150] and [5, pp. 101–102], it is easy to see that $X_{\rho}$ is not reflexive.

To prove that $X_{\rho}$ is of type 2 if $\rho > 2$, it is helpful to prove three lemmas in preparation. The three-dimensional version of Lemma 1 contains essentially the same arguments as used in [5, p. 105]. When $n$ is given, we shall let $A_{\sigma}$ denote the average over the $2^n$ possible arrangements $\sigma$ of $n$ consecutive signs.

LEMMA 1. *For each $n$, a sufficient condition that*

$$A_{\sigma}\left\|\left(\sum_{j=1}^{n} \pm x^j\right)_{\sigma}\right\| < K, \tag{3}$$

*for any choice of $\{x^j : 1 \leqq j \leqq n\}$ in $X_{\rho}$ with each $\|x^j\| = 1$, is that there exist nonnegative numbers $\{\phi_i(n) : i \leqq N\}$ which have the two properties:*

(i) $\sum_{i=1}^{N} [n\phi_i(n)]^{1/\rho} < K$;

(ii) *if each $\xi^j$, $1 \leqq j \leqq n$, is the sum of $r_j$ disjoint bumps of altitudes $+1$ or $-1$ with each $r_j \geqq 0$, then it is possible to have, for each $\sigma$,*

$$\left(\sum_{j=1}^{n} \pm \xi^j\right)_{\sigma} = \sum_{i=1}^{N} \xi^i_{\sigma},$$

*where each $\xi^i_{\sigma}$ is the sum of disjoint bumps of altitudes $+1$ or $-1$ and, if $\bar{r}_i$ is the average over $\sigma$ of the number of bumps in $\xi^i_{\sigma}$, then*

$$\bar{r}_i \leqq \phi_i(n) \sum_{j=1}^{n} r_j \quad \textit{for each } i. \tag{4}$$

PROOF. By the same arguments used in [5, pp. 102–103], it is sufficient to establish (3) with $\|\ \|$ replaced by $\llbracket\ \rrbracket$ and $\llbracket x^j \rrbracket = 1$ for each $j$. As noted in [5, pp. 104–105], it follows from the telescoping nature of the bracketed terms in (2) that there exist numbers $m$ and $\{h_{\mu} : 1 \leqq \mu \leqq m\}$ such that, for each $x^j$, there exists a finite sequence of non-negative integers $\{r_{\mu j} : 1 \leqq \mu \leqq m\}$ such that

$$1 = [\![x^j]\!]^p = \sum_{\mu=1}^{m} r_{\mu j}\left[\left(\sum_{k=\mu}^{m} h_k\right)^p - \left(\sum_{k=\mu+1}^{m} h_k\right)^p\right],$$

where $x^j = \sum_{\mu=1}^{m} \xi_\mu^j$ and each $\xi_\mu^j$ is the sum of $r_{\mu j}$ disjoint bumps whose altitudes have absolute values $h_\mu$. Now use (ii) and obtain, for each $\mu$ and $\sigma$,

$$\left(\sum_{j=1}^{n} \pm \xi_\mu^j\right)_\sigma = \sum_{i=1}^{N} \xi_{\mu\sigma}^i,$$

where each $\xi_{\mu\sigma}^i$ is the sum of $r_{\mu\sigma}^i$ disjoint bumps whose altitudes have absolute values $h_\mu$, and

$$\bar{r}_\mu^i \leqq \phi_i(n)\sum_{j=1}^{n} r_{\mu j} \quad \text{for each } \mu \text{ and } j, \tag{5}$$

where $\bar{r}_\mu^i$ is the average over $\sigma$ of $r_{\mu\sigma}^i$. For each $\sigma$, we have

$$\left(\sum_{j=1}^{n} \pm x^j\right)_\sigma = \left[\sum_{j=1}^{n} \pm\left(\sum_{\mu=1}^{m} \xi_\mu^j\right)\right]_\sigma = \sum_{\mu=1}^{m}\left(\sum_{j=1}^{n} \pm \xi_\mu^j\right)_\sigma$$

$$= \sum_{\mu=1}^{m}\left(\sum_{i=1}^{N} \xi_{\mu\sigma}^i\right) = \sum_{i=1}^{N}\left(\sum_{\mu=1}^{m} \xi_{\mu\sigma}^i\right).$$

Thus it follows from the triangle inequality and convexity of $t^p$ that

$$\begin{aligned} A_\sigma\left\|\left(\sum_{j=1}^{n} \pm x^j\right)_\sigma\right\| &\leqq \sum_{i=1}^{N} A_\sigma\left\|\sum_{\mu=1}^{m} \xi_{\mu\sigma}^i\right\| \\ &\leqq \sum_{i=1}^{N}\left\{A_\sigma\left\|\sum_{\mu=1}^{m} \xi_{\mu\sigma}^i\right\|^p\right\}^{1/p}. \end{aligned} \tag{6}$$

Since, for each $i$, we have

$$\left\|\sum_{\mu=1}^{m} \xi_{\mu\sigma}^i\right\|^p \leqq \sum_{\mu=1}^{m} r_{\mu\sigma}^i\left[\left(\sum_{k=\mu}^{m} h_k\right)^p - \left(\sum_{k=\mu+1}^{m} h_k\right)^p\right],$$

it follows from (6) and (5) that

$$\begin{aligned} A_\sigma\left\|\left(\sum_{j=1}^{n} \pm x^j\right)_\sigma\right\| &\leqq \sum_{i=1}^{N}\left\{\sum_{\mu=1}^{m} \bar{r}_\mu^i\left[\left(\sum_{k=\mu}^{m} h_k\right)^p - \left(\sum_{k=\mu+1}^{m} h_k\right)^p\right]\right\}^{1/p} \\ &\leqq \sum_{i=1}^{N}\left\{\sum_{\mu=1}^{m}\left[\phi_i(n)\sum_{j=1}^{n} r_{\mu j}\right]\left[\left(\sum_{k=\mu}^{m} h_k\right)^p - \left(\sum_{k=\mu+1}^{m} h_k\right)^p\right]\right\}^{1/p} \\ &\leqq \sum_{i=1}^{N}\left\{\phi_i(n)\sum_{j=1}^{n}\sum_{\mu=1}^{m} r_{\mu j}\left[\left(\sum_{k=\mu}^{m} h_k\right)^p - \left(\sum_{k=\mu+1}^{m} h_k\right)^p\right]\right\}^{1/p} \\ &= \sum_{i=1}^{N}\left\{\phi_i(n)\sum_{j=1}^{n} [\![x^j]\!]^p\right\}^{1/p} \\ &= \sum_{i=1}^{N}[n\phi_i(n)]^{1/p} < K. \end{aligned}$$

The following inequality (7) is needed for Lemma 2. Let

$$P_k^n = 2^{-n}\sum_{i=0}^{k}\binom{n}{i}.$$

By using the equality $P_k^n = \frac{1}{4}(P_{k-2}^{n-2} + 2P_{k-1}^{n-2} + P_k^{n-2})$, it is not difficult to prove for $n \geqq 5$ that

$$P_k^n < \tfrac{1}{2}e^{-(n-2k)^2(2n)^{-1}} \quad \text{if} \quad k < \begin{cases} \frac{1}{2}n, & \text{when } n \text{ is even,} \\ \frac{1}{2}(n-1), & \text{when } n \text{ is odd.} \end{cases}$$

If we let $n - 2k = \kappa$ and let $\varepsilon$ be 1 or 2 according as $n$ is odd or even, then for any positive $\rho$ and $n \geqq 5$, we have

$$\begin{aligned} \sum_{k=0}^{[(1/2)(n-1)]} (P_k^n)^{1/\rho} &< \sum_{k=0}^{[(1/2)(n-1)]} 2^{-1/\rho}e^{-(n-2k)^2(2n\rho)^{-1}} \\ &= \sum_{\kappa=\varepsilon}^{n} 2^{-1/\rho}e^{-\kappa^2(2n\rho)^{-1}} < 2^{-1/\rho}\int_0^\infty e^{-x^2(2n\rho)^{-1}}dx \qquad (7) \\ &= \left(\frac{1}{2}\pi\rho\right)^{1/2} 2^{-1/\rho}n^{1/2}, \end{aligned}$$

where the error in approximating $P_k^n$ when $n$ is odd and $k = \frac{1}{2}(n-1)$ is more than balanced by the integral approximation. If $n = 4$, inequality (7) becomes $(\frac{1}{16})^{1/\rho} + (\frac{5}{16})^{1/\rho} < (\frac{1}{2}\pi\rho)^{1/2}2^{-1/\rho}2$, or $(\frac{1}{8})^{1/\rho} + (\frac{5}{8})^{1/\rho} < 2(\frac{1}{2}\pi\rho)^{1/2}$, which clearly is true if $\rho \geqq 2$.

LEMMA 2. *Suppose $C$ and $\{\Delta(n): n \geqq 1\}$ are positive numbers such that, for each $n$, there exist positive numbers $\{\phi_i(n): 1 \leqq i \leqq N(n)\}$ for which*

$$\sum_{i=1}^{N(n)} [n\phi_i(n)]^{1/\rho} \leqq Cn^{\frac{1}{2}+\Delta(n)}, \qquad (8)$$

*and* (ii) *of Lemma* 1 *is satisfied for $X_\rho$. Then for each $n > 4$ and each $\alpha$ with $0 \leqq \alpha < 1$ for which $n^\alpha$ is an integer, there are positive numbers $\{\phi_i'(n): 1 \leqq i \leqq N'(n)\}$ such that* (ii) *of Lemma* 1 *is satisfied for $X_\rho$, and*

$$\sum_{i=1}^{N'(n)} [n\phi_i'(n)]^{1/\rho} < 3^{1/\rho}Cn^{\alpha[\frac{1}{2}+\Delta(n^\alpha)]+(1-\alpha)/\rho} + 6^{1/\rho}\left(\frac{1}{2}\pi\rho\right)^{1/2} n^{\frac{1}{2}+(1-\alpha)/\rho}. \qquad (9)$$

PROOF. Let each $\xi^j$, $1 \leqq j \leqq n$, be the sum of $r_j$ disjoint bumps with altitudes $+1$ or $-1$, where each $r_j \geqq 0$. The norm for $X_\rho$ is *repetition-invariant*, meaning that $\|x\|$ depends only on the distinct numbers used as components of $x$ and their order, but is independent of repetitions of a number. Suppose two bumps have a common endpoint and one bump is stretched or shrunk to the next integer and this integer is not the endpoint of any bump. When representing $(\Sigma_j \pm \xi^j)_\sigma$ as $\Sigma_i \xi^i_\sigma$, this cannot decrease the number of bumps needed in the various $\xi^i_\sigma$'s. Thus there is no loss of generality in proving Lemma 2 with the assumption that no two bumps among all those involved in the various $\xi^j$'s have a common endpoint. Since there are then $2\Sigma_{j=1}^n r_j$ endpoints, there is an interval $I$ which contains the support of each $\xi^j$ and is the union of $(2\Sigma_{j=1}^n r_j) - 1$ intervals on each of which each $\xi^j$ is constant.

For an arbitrary $\alpha$ for which $0 \leqq \alpha < 1$ and $n^\alpha$ is an integer, partition $I$ into intervals $\{I_k\}$ such that each $I_k$ is the union of at most $n^\alpha$ consecutive intervals on which each $\xi^j$ is constant. If $n^\alpha < \Sigma_{j=1}^n r_j$, we can have

$$(10) \qquad 1 \leqq k < 1 + \frac{2\sum_{j=1}^n r_j}{n^\alpha} < \frac{3\sum_{j=1}^n r_j}{n^\alpha}.$$

If $n^\alpha \geqq \Sigma_{j=1}^n r_j$, let the partition consist only of $I$ itself. Now choose two sets $A$ and $B$ of vectors as follows. If $n^\alpha \geqq \Sigma_{j=1}^n r_j$, let $A = \{\xi^j\}$ and $B = \varnothing$. If $n^\alpha < \Sigma_{j=1}^n r_j$, we define, for each vector $\xi^j$, a vector $\eta^j$ in $A$ and $\zeta^j$ in $B$. For each $j$, let $\eta^j$ be the sum of all bumps with the property that each bump either is a bump of $\xi^j$ whose support is a proper subset of some $I_k$, or it is the part of a bump of $\xi^j$ that extends into but not across some $I_k$, i.e., if $I_k$ is not contained in the support of $\xi^j$, then $\xi^j$ and $\eta^j$ have the same intersections with the characteristic function of $I_k$. Then let $\zeta^j = \xi^j - \eta^j$. Note that either $\eta^j$ or $\zeta^j$ or both might be 0.

Since no two bumps involved in the various $\xi^j$'s have a common endpoint, and each bump of an $\eta^j$ that has a point of support in $I_k$ is constant on at least one subinterval of $I_k$, at most $n^\alpha$ vectors in $A$ have their supports in the same $I_k$ and those with support in $I_k$ have a total of at most $n^\alpha$ bumps. Therefore, we can have

$$\left(\sum_{j=1}^n \pm \eta^j_k\right)_\sigma = \sum_{i=1}^{N(n^\alpha)} \eta^i_{k\sigma} \quad \text{for each } k,$$

where $\eta^j_k$ is the restriction of $\eta^j$ to $I_k$, and also have

$$(11) \qquad \bar{s}^i_k \leqq \phi_i(n^\alpha)n^\alpha, \quad \text{if} \quad n^\alpha < \sum_{j=1}^n r_j,$$

and

(12) $$\bar{s}_k^i \leqq \phi_i(n^\alpha)\sum_{j=1}^n r_j, \quad \text{if} \quad n^\alpha \geqq \sum_{j=1}^n r_j,$$

where $\bar{s}_k^i$ is the average over $\sigma$ of the number of bumps in $\eta_{k\sigma}^i$. Now let $\eta_\sigma^i = \Sigma_k \eta_{k\sigma}^i$ for each $i$. Then

$$\left(\sum_{j=1}^n \pm \eta^j\right)_\sigma = \sum_{i=1}^{N(n^\alpha)} \eta_\sigma^i.$$

If $\bar{s}_i$ is the average over $\sigma$ of the number of bumps in $\eta_\sigma^i$, then it follows from (10) and (11) that

(13) $$\bar{s}_i \leqq 3\phi_i(n^\alpha)n^\alpha \frac{\sum_{j=1}^n r_j}{n^\alpha} = 3\phi_i(n^\alpha)\sum_{j=1}^n r_j,$$

if $n^\alpha < \Sigma_{j=1}^n r_j$. Because of (12), inequality (13) is valid when $n^\alpha \geqq \Sigma_{j=1}^n r_j$, even if the "3" is deleted.

Each of the $n$ or fewer vectors in $B$ is the sum of bumps of altitudes $+1$, or $-1$, each having as support a union of consecutive $I_k$'s. We will now choose $\{\zeta_\sigma^i\}$ so that, for each $\sigma$,

(14) $$\left(\sum_{j=1}^n \pm \zeta^j\right)_\sigma = \sum_{i=1}^n \zeta_\sigma^i,$$

with each $\zeta_\sigma^i$ the sum of bumps with altitudes $+1$ or $-1$. For $1 \leqq i \leqq n$, let $\zeta_\sigma^i$ have value 0 in all intervals $I_k$ for which $|(\Sigma_{j=1}^n \pm \zeta^j)_\sigma| < n - i + 1$, and otherwise let $\zeta_\sigma^i$ have value $+1$ or $-1$ according as $(\Sigma_{i=1}^n \pm \zeta^i)_\sigma$ is positive or negative. Then (14) is satisfied.

If exactly $m$ of the $\zeta^j$'s are nonzero on $I_k$, then the number of arrangements of signs $\sigma$ for which $\zeta_\sigma^i$ is nonzero on $I_k$ is $2^n$ times the probability that $n - i + 1$ is less than or equal to the absolute value of the difference between the number of successes and the number of failures in $m$ Bernoulli events with probability $\frac{1}{2}$. This probability does not decrease by more than $\frac{1}{2}$ if $m$ is replaced by $n$. Thus, if $\bar{t}_i$ is the average over $\sigma$ of the number of bumps in $\zeta_\sigma^i$, then it follows from (10) that

(15) $$\bar{t}_i < 4\frac{\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{[\frac{1}{2}(i-1)]}}{2^n} \cdot \frac{3\sum_{j=1}^n r_j}{n^\alpha}.$$

We now have $(\Sigma_{j=1}^n \pm x^j)_\sigma = \Sigma_{i=1}^{N(n^\alpha)} \eta_\sigma^i + \Sigma_{i=1}^n \zeta_\sigma^i$. Also, there are $N(n^\alpha) + n$ new $\phi_i$'s, which we denote by $\{\phi_i'(n)\colon 1 \leqq i \leqq N(n^\alpha) + n\}$, and choose by use of (13) and (15) so as to satisfy (4). Then

$$\sum_i [n\cdot\phi_i'(n)]^{1/p} \leqq \sum_{i=1}^{N(n^\alpha)} [3n\cdot\phi_i(n^\alpha)]^{1/p} + \sum_{i=1}^{n}\left[12n\frac{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{[\frac{1}{2}(i-1)]}}{2^n n^\alpha}\right]^{1/p},$$

$$= 3^{1/p}n^{(1-\alpha)/p}\sum_{i=1}^{N(n^\alpha)}[n^\alpha\phi_i(n^\alpha)]^{1/p} + 12^{1/p}n^{(1-\alpha)/p}\sum_{i=1}^{n}\left[\frac{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{[\frac{1}{2}(i-1)]}}{2^n}\right]^{1/p}.$$

The first of these summations (without its coefficient) is by hypothesis not greater than $Cn^{\alpha[\frac{1}{2}+\Delta(n^\alpha)]}$. The second summation is less than $(2\pi p)^{\frac{1}{2}}2^{-1/p}n^{\frac{1}{2}}$ if $n \geqq 4$, since it is not greater than $2\Sigma_{k=0}^{[1/2(n-1)]}(P_k^n)^{1/p}$, which (7) implies is less than $(2\pi p)^{\frac{1}{2}}2^{-1/p}n^{\frac{1}{2}}$. Thus

$$\sum_i [n\phi_i'(n)]^{1/p} < 3^{1/p}Cn^{\alpha[\frac{1}{2}+\Delta(n^\alpha)]+(1-\alpha)/p} + 6^{1/p}(2\pi p)^{\frac{1}{2}}n^{\frac{1}{2}+(1-\alpha)/p}.$$

LEMMA 3. *Let $n > 4$ be a positive integer. Suppose $C$ and $\{\Delta(k): 1 \leqq k < n\}$ are nonnegative numbers, and that there exist nonnegative numbers $\{\phi_i(k)\}$ for each $k < n$ which can be used in Lemma 1 to verify that, if $\|x^j\| = 1$ for each $j$, then*

$$A_\sigma\left\|\left(\sum_{j=1}^{k} \pm x^j\right)_\sigma\right\| \leqq Ck^{\frac{1}{2}+\Delta(k)}. \tag{16}$$

*If $0 \leqq \alpha < 1$ and $n^\alpha$ is an integer, then there exist nonnegative numbers $\{\phi_i'(n)\}$ which can be used in Lemma 1 to verify that, if $\|x^j\| = 1$ for each $j$, then*

$$A_\sigma\left\|\left(\sum_{j=1}^{n} \pm x^j\right)_\sigma\right\| < 3^{1/p}Cn^{\alpha[\frac{1}{2}+\Delta(n^\alpha)]+(1-\alpha)/p} + 6^{1/p}(2\pi p)^{\frac{1}{2}}n^{\frac{1}{2}+(1-\alpha)/p}. \tag{17}$$

PROOF. This Lemma is an immediate consequence of Lemmas 1 and 2.

THEOREM. *The space $X_p$ is of type 2 if $p > 2$. Moreover, for each $n$ and any $\{x^j: 1 \leqq j \leqq n\}$ in $X_p$ with $\|x^j\| = 1$ for each $j$, we have*

$$A_\sigma\left\|\left(\sum_{j=1}^{n} \pm x^j\right)_\sigma\right\| \leqq [3^{(p+1)/(p-2)}(2e)^{1/p}(\tfrac{1}{2}\pi p)^{\frac{1}{2}}]n^{\frac{1}{2}}. \tag{18}$$

*For $X_2$ and any $\theta > 0$, there is an $n_\theta$ such that*

$$A_\sigma \left\| \left( \sum_{j=1}^{n} \pm x^j \right)_\sigma \right\| \leqq \begin{cases} n_\theta^{1/2} n^{\frac{1}{2}}, & \text{if } n \leqq n_\theta \\ 6(3en_\theta)^{\frac{1}{2}} n^{\frac{1}{2}+\theta/[\ln(\ln n)]}, & \text{if } n > n_\theta. \end{cases} \tag{19}$$

PROOF. Assume first that $\rho > 2$. Introduce the functions $\beta$ and $\Delta$ defined for $n > 3$ by

$$\beta(n) = \frac{(1+\rho)\ln 3}{\rho \ln n} \quad \text{or} \quad n^{\beta(n)} = 3^{(1+\rho)/\rho}, \tag{20}$$

$$\Delta(n) = \frac{\rho}{\rho - 2}\beta(n) + \frac{1}{\rho \ln n}, \tag{21}$$

and $\beta(n) = \Delta(n) = 0$ if $n \leqq 3$. We will show that if $n \geqq 4$ and if $x^j \in X_\rho$ with $\|x^j\| = 1$ for $1 \leqq j \leqq n$, then

$$A_\sigma \left\| \left( \sum_{j=1}^{n} \pm x^j \right)_\sigma \right\| < 2^{1/\rho}(\tfrac{1}{2}\pi\rho)^{\frac{1}{2}} n^{\frac{1}{2}+\Delta(n)}. \tag{22}$$

Since $n^{\Delta(n)} = 3^{(\rho+1)/(\rho-2)} e^{1/\rho}$, this will establish (18) for $n \geqq 4$. Since $\|(\Sigma_1^n \pm x^j)_\sigma\| \leqq n$ if $\|x^j\| = 1$ for each $j$, and the right member of (18) is greater than $3n^{\frac{1}{2}}$ if $\rho > 2$, we see that (18) is valid if $n < 4$.

We will establish (22) for $n > 4$ by induction. In order to use Lemma 3 in the first step of the induction, we need to know that there exist $\{\phi_i(n)\}$ for $n \leqq 4$ which can be used in Lemma 1 to establish (22) for $n \leqq 4$. If we let $\phi_i(n) = 1$ for $1 \leqq i \leqq n$ and $n = 1, 2$, or 4, and let $\phi_i(3) = i^{-1}$ for $1 \leqq i \leqq 3$, then (ii) of Lemma 1 is satisfied and we need to have, for $n \leqq 4$ and $\rho > 2$,

$$\sum_{i=1}^{n} [n\phi_i(n)]^{1/\rho} \leqq 2^{1/\rho}(\tfrac{1}{2}\pi\rho)^{\frac{1}{2}} n^{\frac{1}{2}+\Delta(n)}. \tag{23}$$

This is satisfied for $n = 1$, since $1 < 2^{1/\rho}(\frac{1}{2}\pi\rho)^{\frac{1}{2}}$; for $n = 2$, since $2 \cdot 2^{1/\rho} < 2^{1/\rho}(\frac{1}{2}\pi\rho)^{\frac{1}{2}} 2^{\frac{1}{2}}$; for $n = 3$, since $3^{1/\rho}(1 + 2^{-1/\rho} + 3^{-1/\rho}) < 2^{1/\rho}(\frac{1}{2}\pi\rho)^{\frac{1}{2}} 3^{\frac{1}{2}}$ follows from $1 + 2^{-1/\rho} + 3^{-1/\rho} < 2^{1/\rho}(\frac{1}{2}\pi\rho)^{\frac{1}{2}}$; and for $n = 4$, since $4 \cdot 4^{1/\rho} < 3^{(\rho+1)/(\rho-2)}(2e)^{1/\rho}(\frac{1}{2}\pi\rho)^{\frac{1}{2}} 2$. Now suppose $n > 4$ and for each $k < n$ there exist nonnegative numbers $\{\phi_i(k)\}$ which can be used in Lemma 1 to verify that (22) is valid if $n$ is replaced by any positive integer $k < n$. The principal technique in the proof is to choose $C$ in (16) by letting

$$C = 2^{1/\rho}(\tfrac{1}{2}\pi\rho)^{\frac{1}{2}}, \tag{24}$$

and then using Lemma 3 after making each exponent of $n$ in (17) less than or equal to

(25) $$\tfrac{1}{2}+\Delta(n)-\beta(n).$$

This will imply that (22) is satisfied, since $n^{-\beta(n)}=3^{-1-1/\rho}$, and

$$3^{-1-1/\rho}[3^{1/\rho}C+6^{1/\rho}(2\pi\rho)^{\frac{1}{2}}]=C.$$

For the second exponent of $n$ in (17) to be less than or equal to (25), we need

(26) $$\alpha \geqq 1+\rho[\beta(n)-\Delta(n)].$$

For the first exponent of $n$ in (17) to be less than or equal to (25), we need

(27) $$\frac{1}{2}\alpha+\frac{1-\alpha}{\rho}\leqq\frac{1}{2}+\Delta(n)-\beta(n) \quad \text{if} \quad n^{\alpha}\leqq 3,$$

and

(28) $$\alpha\left[\frac{1}{2}+\Delta(n^{\alpha})\right]+\frac{1-\alpha}{\rho}\leqq\frac{1}{2}+\Delta(n)-\beta(n) \quad \text{if} \quad n^{\alpha}>3.$$

Since $\Delta(n^{\alpha})=\Delta(n)/\alpha$, these are equivalent, respectively, to

(29) $$\alpha\leqq 1+\frac{2\rho}{\rho-2}[\Delta(n)-\beta(n)] \quad \text{if} \quad n^{\alpha}\leqq 3,$$

(30) $$\alpha\leqq 1-\frac{2\rho}{\rho-2}\beta(n) \quad \text{if} \quad n^{\alpha}>3.$$

Since (29) is satisfied if $\alpha<1$, the inductive proof is completed for all $n$ for which $\alpha$ can be chosen so that $n^{\alpha}\leqq 3$ and $\alpha$ satisfies (26). That is, for all $n$ such that

(31) $$n^{1+\rho[\beta(n)-\Delta(n)]}\leqq 3,$$

which is true if $n\cdot n^{-2\rho\beta(n)/(\rho-2)}\leqq 3$. Since $n^{\beta(n)}=3^{(1+\rho)/\rho}$, this is satisfied if

(32) $$n\leqq 3^{3\rho/(\rho-2)}.$$

Now assume that $n>3^{3\rho/(\rho-2)}$. Because

(33) $$n^{x+(\ln n)^{-1}}=e\cdot n^{x}>1+n^{x} \quad \text{if} \quad x\geqq 0,$$

there is an $\alpha$ that satisfies both (26) and (30) and also satisfies $0\leqq\alpha<1$ and the requirement that $n^{\alpha}$ be an integer, if

(34) $$\left[1-\frac{2\rho}{\rho-2}\beta(n)\right]-\{1+\rho[\beta(n)-\Delta(n)]\}\geqq\frac{1}{\ln n},$$

and also

$$(35)\qquad \frac{1}{\ln n} \leqq 1 - \frac{2\rho}{\rho - 2}\beta(n).$$

Because of (21), (34) is an equality. Write (35) as

$$(36)\qquad 1 + \frac{2(1+\rho)\ln 3}{\rho - 2} \leqq \ln n.$$

Since the inequality, $\ln n > \ln 3 + 2(1+\rho)(\rho-2)^{-1}\ln 3$, follows from $n > 3^{3\rho/(\rho-2)}$, (36) is satisfied.

This completes the proof of Theorem 1 for $\rho > 2$. Let us now consider the case $\rho = 2$. For $\theta$ an arbitrary positive number, choose $n_\theta$ so that

$$(37)\qquad \ln(\ln n_\theta) \geqq \max\{4, 4\theta\} \quad \text{and} \quad \frac{[\ln(\ln n_\theta)]^2}{\ln n_\theta} \leqq \frac{2\theta^2}{3\ln 3}.$$

Define $\beta(n)$ as in (20) for $n > n_\theta$, but with $\rho$ replaced by 2. Let

$$(38)\qquad \Delta(n) = \frac{\theta}{\ln(\ln n)} + \beta(n) + \frac{1}{2\ln n}, \quad \text{if} \quad n > n_\theta,$$

and let $\beta(n) = \Delta(n) = 0$ if $n \leqq n_\theta$. Choose $D$ so that, for $n \leqq n_\theta$, there exist $\{\phi_i(n)\}$ which can be used in Lemma 1 to prove that

$$(39)\qquad A_\sigma\left\|\left(\sum_{j=1}^{n} \pm x^j\right)_\sigma\right\| \leqq Dn^{\frac{1}{2}}, \quad \text{if } n \leqq n_\theta \text{ and } \|x^j\| = 1.$$

For example, we could let $\phi_i(n) = i^{-1}$ for $i \leqq n$ and note that

$$n^{\frac{1}{2}}\sum_{1}^{n} i^{-\frac{1}{2}} < 2n,$$

to see that $D$ could be $2n_\theta^{1/2}$. We will show that, if each $x^j$, $1 \leqq j \leqq n$, is in $X_2$ with $\|x^j\| = 1$, then, for all $n > n_\theta$,

$$(40)\qquad A_\sigma\left\|\left(\sum_{j=1}^{n} \pm x^j\right)_\sigma\right\| \leqq Dn^{\frac{1}{2}+\Delta(n)}.$$

Since $n^{\Delta(n)} = 3(3e)^{\frac{1}{2}} \cdot n^{\theta/[\ln(\ln n)]}$, this implies (19) for $n > n_\theta$. We will establish (40) by induction. Suppose $n > n_\theta$ and there exist nonnegative numbers $\{\phi_i(k)\}$ for each $k < n$ which can be used in Lemma 1 to verify (40) if $n$ is replaced by any $k < n$. The principal technique in the proof is to use Lemma 3 after making each exponent of $n$ in (17) less than or equal to

$$(41)\qquad \tfrac{1}{2} + \Delta(n) - \beta(n).$$

This will imply that (40) is satisfied, since $n^{-\beta(n)} = 3^{-3/2}$ and (37) implies

$$3^{-3/2}[3^{\frac{1}{2}}D + 6^{\frac{1}{2}}(4\pi)^{\frac{1}{2}}] < D \quad \text{if} \quad D = 2n_\theta^{1/2}.$$

For the second exponent of $n$ in (17) to be less than or equal to (41), we need

$$\alpha \geqq 1 + 2[\beta(n) - \Delta(n)]. \tag{42}$$

For the first exponent of $n$ in (17) to be less than or equal to (41), we need (27) and (28) with $\rho = 2$ and 3 replaced by $n_\theta$. Since $\Delta(n^\alpha)$ is equal to $\theta/[\ln\alpha + \ln(\ln n)] + [\beta(n) + (2\ln n)^{-1}]/\alpha$, these are equivalent to

$$0 \leqq \Delta(n) - \beta(n), \quad \text{if} \quad n^\alpha \leqq n_\theta, \tag{43}$$

$$\frac{\alpha\theta}{\ln\alpha + \ln(\ln n)} + \beta(n) \leqq \frac{\theta}{\ln(\ln n)}, \quad \text{if} \quad n^\alpha > n_\theta. \tag{44}$$

Since $\ln(\ln n) > 0$, (43) is valid for all $n$. Thus the inductive proof is completed if $\alpha$ can be chosen so that (42) and (44) are satisfied, whether or not $n^\alpha \leqq n_\theta$. Inequality (42) is equivalent to

$$\alpha \geqq 1 - \frac{2\theta}{\ln(\ln n)} - \frac{1}{\ln n},$$

and the right member of this inequality is positive and greater than $\frac{1}{2} - 1/(\ln n)$ because of (37). Thus there is an $\alpha$ with $0 \leqq \alpha < 1$ which satisfies (42), (44) and the requirement that $n^\alpha$ be an integer, if (44) is satisfied for all $\alpha$ between

$$1 - \frac{2\theta}{\ln(\ln n)} - \frac{1}{\ln n} \quad \text{and} \quad 1 - \frac{2\theta}{\ln(\ln n)}.$$

For $\alpha$ in this interval, the left member of (44) increases with $\alpha$, since $\alpha > \frac{1}{2} - 1/(\ln n) > e/(\ln n)$, so it is sufficient to have

$$\frac{\{1 - 2\theta[\ln(\ln n)]^{-1}\}\theta}{-4\theta[\ln(\ln n)]^{-1} + \ln(\ln n)} + \frac{3(\ln 3)}{2(\ln 2)} \leqq \frac{\theta}{\ln(\ln n)}.$$

This can be proved by using the following inequality, which follows from (37):

$$\frac{3(\ln 3)}{2(\ln 2)} < \theta^2\left[\frac{2 - 4[\ln(\ln n)]^{-1}}{[\ln(\ln n)]^2}\right].$$

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